Formulation of finite-time singularity for free-surface Euler equations

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Abstract

We give an extremely short proof that the free-surface incompressible, irrotational Euler equations with regular initial condition can form a finite time singularity in 2D or 3D. Thus, we provide a simple view of the problem studied by Castro et al [5], [6], [7], [8], [9] and Coutand & Shkoller [14].

1 Introduction

The aim of this paper is to give an extremely short proof that the free-surface incompressible, irrotational Euler equations with regular initial conditions can form a finite time singularity in 2D or 3D. Before going further, we make some historical remarks on the problem. For the irrotational case of the water waves problem, the local wellposedness in Sobolev space was established by Wu [22] in 2D and Wu [23] in 3D. Earlier works dealt with small data or linearized equations, see Nalimov [20], Yoshihara [26], Craig [10], Beale et al [4]. Further studies of local wellposedness was done by Ambrose &Masmoudi [2] and by Lannes [18]. For small initial data, Wu [24] established almost global existence for the infinite-depth problem in 2D. Independently, Wu [25] and Germain et al [15], [17] established global existence for the infinite-depth problem in 3D. For more bibliographic notes, we refer to [11], [12], [13], [16], [19], [21], [27].

The problem of establishing a finite-time singularity for the interface has recently been explored for the 2D water waves equations by Castro et al [7], [8], where it was shown that a smooth initial curve exhibits a finite-time singularity via self-intersection at a point. More recently, Coutand and Shkoller [14] established a similar result in 3D for the general incompressible case, and their method can also be used in 2D. Both approaches are to start from a curve which self-intersecting at a point and solve backward to get a regular solution at an early time. In this paper, we start from given initial conditions and show by contradiction that a finite-time singularity must form if the initial data satisfies some positivity condition. Thus, we provide an extremely simple view of the problem. However, the price is to fix the endpoint as will be see from below. Our method works both in 2D and 3 D, with or without gravity, for simplicity, we just give a proof in 2D without gravity. As our proof is extremely simple, it is not hard to figure out a same proof in 3D or with gravity.

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We consider the 2D free-surface incompressible, irrotational Euler equations in $\Omega(t)$, where $\Omega(t)$ is bounded by $x_1 = 0, 1$, and $x_2 = 0$ and a free surface $\Gamma(t)$. We assume that $\Gamma(t)$ is parametered as

$$\Gamma(t) = \{\big(x_1(t,\alpha), x_2(t,\alpha)\big), 0 \le \alpha \le 1\}, \tag{1.1}$$

with

$$x_2(t,0) = x_2(t,1) = 1, \tag{1.2}$$

$$0 \le x_1(t,\alpha) \le 1, 0 \le t < \infty, 0 \le \alpha \le 1. \tag{1.3}$$

Our problem is modeled by the following

$$\begin{cases} u_t^1 + u^1\partial_1 u^1 + u^2\partial_2 u^1 + \partial_1 p = 0, \\ u_t^2 + u^1\partial_1 u^2 + u^2\partial_2 u^1 + \partial_2 p = 0, \quad on\ \Omega(t) \end{cases} \tag{1.4}$$

$$\begin{cases} \partial_1 u^1 + \partial_2 u^2 = 0, \\ \partial_2 u^1 - \partial_1 u^2 = 0, \quad on\ \Omega(t) \end{cases} \tag{1.5}$$

$$\begin{cases} u^1 = 0, \quad x_1 = 0, 1 \\ u^2 = 0, \quad x_2 = 0, \quad on\ \Omega(t) \end{cases} \tag{1.6}$$

$$u^2(t,0,1) = u^2(t,1,1) = 0 \tag{1.7}$$

$$\begin{cases} \dfrac{dx_1(t,\alpha)}{dt} = u^1(t, x_1(t,\alpha), x_2(t,\alpha)), \quad 0 \le \alpha \le 1 \\ \dfrac{dx_2(t,\alpha)}{dt} = u^2(t, x_1(t,\alpha), x_2(t,\alpha)), \end{cases} \tag{1.8}$$

$$t = 0 : u^1 = u_0^1, u^2 = u_0^2 \quad on \quad \Omega(0) \tag{1.9}$$

where u_0^1, u_0^2 satisfies (1.5) and (1.2), (1.3), (1.6), (1.7) at $t = 0$.
Our main result can be summarized as follows:
Theorem 1.1. Consider the free-surface Euler equation (1.1)-(1.9), let $\Gamma(t)$ and u^1, u^2 be a C^1 solution with C^1 initial data $\Gamma(0)$ and (u_0^1, u_0^2), then $\Gamma(t)$ must develop a finite time singularity provided that

$$A = \int_{\Omega(0)} u_0^1 x_1 dx + \int_0^1 x_2 u_0^2(x_2, 1) dx_2 > 0.$$

2 Proof of main result

As observed by Lindblad [11]

$$\begin{aligned} -\Delta p &= (\partial_1 u^1)^2 + (\partial_2 u^2)^2 + 2\partial_1 u^2 \partial_2 u^1 \\ &= (\partial_1 u^1)^2 + (\partial_2 u^2)^2 + 2(\partial_2 u^1)^2 \\ &\ge 0, \end{aligned} \tag{2.1}$$

where $\triangle = \partial_1^2 + \partial_2^2$,

$$p = 0 \quad on\ \Gamma(t),$$

it is not difficult to see from boundary condition that

$$\frac{\partial p}{\partial n} = 0, \ x_1 = 0, 1 \ or \ x_2 = 0, \tag{2.2}$$

where n is outward norm.
Therefore, by maximum principle, we get

$$p \geq 0 \ on \ \Omega(t). \tag{2.3}$$

Let

$$\frac{D}{Dt} = \partial_t + u^1\partial_1 + u^2\partial_2 \tag{2.4}$$

be the material derivative, then

$$\begin{aligned} \frac{D(u^1 x_1)}{Dt} &= \frac{Du_1}{Dt} x_1 + u_1 \frac{Dx_1}{Dt} \\ &= -\partial_1 p \ x_1 + u_1^2. \end{aligned} \tag{2.5}$$

Therefore

$$\begin{aligned} \frac{d}{dt}\int_{\Omega(t)} u^1 x_1 dx &= \int_{\Omega(t)} (u^1)^2 dx - \int_{\Omega(t)} \partial_1 p \ x_1 dx \\ &= \int_{\Omega(t)} (u^1)^2 dx + \int_{\Omega(t)} p dx - \int_0^1 p(t, 1, x_2) dx_2 \end{aligned} \tag{2.6}$$

On $x_1 = 1, u^1 = 0$, so we get

$$\partial_t u^2 + u^2 \partial_2 u^2 + \partial_2 p = 0,$$

multiply by x_2 and make an integration by parts, we get

$$\frac{d}{dt}\int_0^1 u^2(t, x_2, 1) x_2 dx_2 = \frac{1}{2}\int (u^2(t, x_1, 1))^2 dx_2 + \int_0^1 p(t, 1, x_2) dx_2. \tag{2.7}$$

Adding (2.6) and (2.7) together and noting (2.3), we get

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega(t)} u^1 x_1 dx + \int_0^1 u^2(t, x_2, 1) x_2 dx_2 \\ &= \int_{\Omega(t)} (u^1)^2 dx + \frac{1}{2}\int_0^1 (u^2(t, x_2, 1))^2 dx_2 + \int_{\Omega(t)} p dx \\ &\geq \int_{\Omega(t)} (u^1)^2 dx + \frac{1}{2}\int_0^1 (u^2(t, x_2, 1))^2 dx_2. \end{aligned} \tag{2.8}$$

By Schwatz inequality and noting (1.5) and (1.3), we get

$$\begin{aligned} \left(\int_{\Omega(t)} u^1 x_1 dx\right)^2 &\leq \left(\int_{\Omega(t)} (u^1)^2 dx\right)\left(\int_{\Omega(t)} x_1^2 dx\right) \\ &\leq \left(\int_{\Omega(t)} (u^1)^2 dx\right)\int_{\Omega(t)} dx \\ &= \int_{\Omega(t)} (u^1)^2 dx \int_{\Omega(0)} dx. \end{aligned}$$

Similarly

$$
\begin{aligned}
&\Big(\int_0^1 x_2 u^2(t,x_2,1)dx_2\Big)^2 \\
&\qquad \le \left(\int_0^1 x_2^2 dx_2\right)\int_0^1 \left(u^2(t,x_2,1)\right)^2 dx_2 \\
&\qquad = \frac{1}{3}\int_0^1 \left(u^2(t,x_2,1)\right)^2 dx_2.
\end{aligned}
$$

Let

$$c_1 = max\big(2\int_{\Omega(0)} dx,\ \frac{4}{3}\big) \tag{2.9}$$

and

$$L(t) = \int_{\Omega(t)} u^1 x_1 dx + \int_0^1 x_2 u^2(t,x_2,1)dx_2 \tag{2.10}$$

then it is not difficult to see

$$\begin{cases} L'(t) \ge \dfrac{L(t)^2}{c_1} \\ L(0) = A > 0 \end{cases} \tag{2.11}$$

Thus, $L(t)$ becomes infinite in finite time.

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